1029. Proposed by José Luis Díaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain and
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Let $\{a_n\}$ be the sequence of real numbers defined by $a_0 = 3$ and $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$ for all nonnegative integers n. Prove that, for all positive integers n,

$$1 + 5\left(\sum_{k=0}^{n} \sqrt{\frac{\binom{n}{k} F_k F_{n-k}}{1 + a_k}}\right)^2 \le 2^{n-1} L_n,$$

where L_n denotes the n th Lucas number, $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for integers $n \ge 2$, and F_n denotes the nth Fibonacci number, $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for integers $n \ge 2$.

Solution by Arkady Alt , San Jose ,California, USA.

First note that by Cauchy Inequality

$$\left(\sum_{k=0}^{n} \sqrt{\frac{\binom{n}{k} F_k F_{n-k}}{1 + a_k}}\right)^2 \le \sum_{k=0}^{n} \binom{n}{k} F_k F_{n-k} \cdot \sum_{k=0}^{n} \frac{1}{1 + a_k}.$$

Since $F_n = \dfrac{\phi^n - \overline{\phi}^n}{\sqrt{5}}$ and $L_n = \phi^n + \overline{\phi}^n, n \in \mathbb{N} \cup \{0\}$, where $\phi = \dfrac{1 + \sqrt{5}}{2}, \overline{\phi} = \dfrac{1 - \sqrt{5}}{2}$,

then $5\sum_{k=0}^{n} \binom{n}{k} F_k F_{n-k} = \sum_{k=0}^{n} \binom{n}{k}\left(\phi^k - \overline{\phi}^k\right)\left(\phi^{n-k} - \overline{\phi}^{n-k}\right) = \sum_{k=0}^{n} \binom{n}{k}(\phi^n + \overline{\phi}^n) - \sum_{k=0}^{n} \binom{n}{k}\phi^k \overline{\phi}^{n-k}$
$\sum_{k=0}^{n} \binom{n}{k}\overline{\phi}^k \phi^{n-k} = 2^n L_n - 2(\phi + \overline{\phi})^n = 2^n L_n - 2.$

Now we should pay attention to $S_n := \sum_{k=0}^{n} \dfrac{1}{1 + a_k}$.

First terms of sequence $\{a_n\}$ are $a_0 = 3, a_1 = 5, a_2 = 13, a_3 = 85, a_4 = 3613$.

We can see that $\frac{1}{2} - S_0 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0, \frac{1}{2} - S_1 = \frac{1}{2} - \left(\frac{1}{4} + \frac{1}{6}\right) = \frac{1}{12} = \frac{1}{a_2 - 1} > 0$, $\frac{1}{2} - S_2 = \frac{1}{2} - \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{14}\right) = \frac{1}{84} = \frac{1}{a_3 - 1} > 0$,
$\frac{1}{2} - S_3 = \frac{1}{2} - \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{14} + \frac{1}{86}\right) = \frac{1}{3612} = \frac{1}{a_4 - 1} > 0.$

We will prove (by Math Induction) that for any $n \ge 1$ holds $\frac{1}{2} - S_n = \frac{1}{a_{n+1} - 1}$.

Since $a_{n+1} - 1 = \frac{1}{2}(a_n^2 + 1) - 1 \Leftrightarrow a_{n+1} - 1 = \frac{a_n^2 - 1}{2}, n \in \mathbb{N} \cup \{0\}$ then, assuming $\frac{1}{2} - S_n = \frac{1}{a_{n+1} - 1}$, we obtain $\frac{1}{2} - S_{n+1} = \frac{1}{a_{n+1} - 1} - \frac{1}{a_{n+1} + 1} = \frac{2}{a_{n+1}^2 - 1} = \frac{1}{a_{n+2} - 1}$.

Thus, $S_n < \frac{1}{2}, n \in \mathbb{N}$ and, therefore,

$$1 + 5\left(\sum_{k=0}^{n} \sqrt{\frac{\binom{n}{k} F_k F_{n-k}}{1 + a_k}}\right)^2 \le 1 + 5\sum_{k=0}^{n} \binom{n}{k} F_k F_{n-k} \cdot \sum_{k=0}^{n} \frac{1}{1 + a_k} =$$

$$1 + (2^n L_n - 2)S_n < 1 + (2^n L_n - 2)\frac{1}{2} = 2^{n-1} L_n.$$